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A new multi-fermion realization of $SU_q(2)$ and its application to the Lipkin–Meshkov–Glick model

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Abstract. A new multi-fermion realization of $SU_q(2)$ is built, whose generators have been written as functions of operators of su(2) algebra. With the help of the realization, the influence of the q-deformation of su(2) algebra on the excitation spectrum and the ground state properties of the Lipkin–Meshkov–Glick model can easily be understood. It is also shown that the variation approach on the basis of the usual su(2) coherent state still works well for the q-deformed model.

Quantum group [1, 2], deformed Lie algebra, has attracted much attention of both physicists and mathematicians in recent years. It has been developed in many respects. In order to clarify its interpretations and possible applications in physics, the study of q-deformed physical systems is significant [3–9] and some solvable models which have the common feature that their Hamiltonians can be written as functions of generators of quantum algebras have been investigated [10–15]. It is very useful when studying these physical systems to express generators of a quantum algebra in terms of operators of the corresponding Lie algebra. In our opinion, this has at least two advantages. One is that we can use the well developed calculation skills of Lie algebras and various approximation approaches for the usual quantum systems. Another is that we can see what is added to an original quantum system through q-deformation and then understand why q-deformation brings about such physical effects. In this paper, we shall employ the Lipkin–Meshkov–Glick (LMG) model [16, 17] to investigate the above idea.

The model under consideration has two energy levels separated by ε. The degeneracy of each level is N. Two quantum numbers p and σ serve to label single particle states, in which p takes integral values from 1 to N, and $\sigma = +1, -1$ are for the upper and lower levels, respectively. The Hamiltonian of the model is

$$\hat{H} = \varepsilon \hat{L}_z - \frac{V}{2}(\hat{L}_+^2 + \hat{L}_-^2) \tag{1}$$

with

$$\hat{L}_z = \tfrac{1}{2}\sum_{\sigma=\pm}\sum_{p=1}^{N} \sigma a_{p\sigma}^+ a_{p\sigma} \qquad \hat{L}_+ = \sum_{p=1}^{N} a_{p+}^+ a_{p-} \qquad \hat{L}_- = (\hat{L}_+)^+ \tag{2}$$

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where $a^+_{p\sigma}(a_{p\sigma})$ creates (annihilates) a particle in the single particle state (p,σ) and V is a constant standing for the strength of the interaction. Since $a^+_{p\sigma}$ and $a_{p\sigma}$ fulfil the anti-commutation relations of fermions, it is easily shown that the commutation relations of $\hat{L}_z$ and $\hat{L}_\pm$ are

$$[\hat{L}_z, \hat{L}_\pm] = \pm\hat{L}_\pm \qquad [\hat{L}_+, \hat{L}_-] = 2\hat{L}_z. \tag{3}$$

Therefore, $\hat{L}_z$ and $\hat{L}_\pm$ constitute a su(2) algebra.

Let $\hat{J}_z$ and $\hat{J}_\pm$ represent the generators of quantum algebra $\mathrm{SU}_q(2)$. It is well known that their commutators are

$$[\hat{J}_z, \hat{J}_\pm] = \pm\hat{J}_\pm \qquad [\hat{J}_+, \hat{J}_-] = [2\hat{J}_z] \tag{4}$$

where

$$[2\hat{J}_z] = \frac{q^{2\hat{J}_z} - q^{-2\hat{J}_z}}{q - q^{-1}}. \tag{5}$$

By setting $q = \exp(\eta)$, we can write (5) as

$$[2\hat{J}_z] = \frac{\sinh(2\eta\hat{J}_z)}{\sinh(\eta)}. \tag{6}$$

Replacing $\hat{L}_z$ and $\hat{L}_\pm$ in (1) by $\hat{J}_z$ and $\hat{J}_\pm$, one can obtain the q-deformed LMG model, whose Hamiltonian is

$$\hat{H} = \varepsilon\hat{J}_z - \frac{V}{2}(\hat{J}_+^2 + \hat{J}_-^2). \tag{7}$$

This model was employed in [11] to test the validity of the variation approach on the basis of the $\mathrm{SU}_q(2)$ coherent state and to study the influence of q-deformation of the su(2) algebra on the ground state phase transition. Since all their calculations were directly carried out in the q-deformed representation regardless of concrete realizations of $\hat{J}_z$ and $\hat{J}_\pm$, it is impossible through that study to understand why q-deformation causes the observed physical phenomena. A multi-fermion realization of the quantum algebra $\mathrm{SU}_q(2)$ was built in [12]. The generator $\hat{J}_+$ ($\hat{J}_-$) in that realization was written as the linear superposition of the original fermion operators $a^+_{p+}a_{p-}(a^+_{p-}a_{p+})$. In this way, particles with different quantum numbers p are treated in different ways and the symmetry among particles which exists in the original model is destroyed in those generators. In the present study, we shall express $\hat{J}_z$ and $\hat{J}_\pm$ as functions of $\hat{L}_z$ and $\hat{L}_\pm$ to overcome the above shortcomings.

We notice that the eigenvalues of $\hat{L}_z$ are differences between the particle numbers of the upper and lower levels. It is reasonable in physics to take

$$\hat{J}_z = \hat{L}_z. \tag{8}$$

Furthermore, considering the first relations of (3) and (4), we set

$$\hat{J}_+ = \hat{L}_+\phi(\hat{L}, \hat{L}_z) \qquad \hat{J}_- = \hat{L}_-\psi(\hat{L}, \hat{L}_z) \tag{9}$$

where ϕ and ψ are functions of $\hat{L}_z$ and $\hat{L}$ defined according to $\hat{L}\cdot\hat{L} = \hat{L}^2$, which stands for the Casimir operator of the Lie algebra su(2). Using commutation relations (3), one easily verifies that the generators constructed as (8) and (9) obey the first commutation relation of (4) identically. The second commutation relation of (4) requires that the operator functions ϕ and ψ satisfy

$$[2\hat{L}_z] = (\hat{L}^2 - \hat{L}_z^2 + \hat{L}_z)\phi(\hat{L}, \hat{L}_z - 1)\psi(\hat{L}, \hat{L}_z) - (\hat{L}^2 - \hat{L}_z^2 - \hat{L}_z)\psi(\hat{L}, \hat{L}_z + 1)\phi(\hat{L}, \hat{L}_z). \tag{10}$$

Since $\hat{J}_\pm$ must be Hermitian conjugate to one another, i.e. $(\hat{J}_+)^+ = \hat{J}_-$ and $(\hat{J}_-)^+ = \hat{J}_+$, it follows that

$$\phi(\hat{L}, \hat{L}_z) = \psi(\hat{L}, \hat{L}_z + 1). \tag{11}$$

Considering $\hat{J}_\pm|_{\eta\to 0} = \hat{L}_\pm$ and using (10) and (11), we find

$$\hat{J}_z = \hat{L}_z \tag{12}$$

$$\hat{J}_+ = \hat{L}_+ \frac{[\sinh^2(\eta\hat{L}) - \sinh(\eta\hat{L}_z)\sinh(\eta(\hat{L}_z+1))]^{1/2}}{[\hat{L}^2 - \hat{L}_z^2 - \hat{L}_z]^{1/2}\sinh(\eta)} \tag{13}$$

$$\hat{J}_- = \hat{L}_- \frac{[\sinh^2(\eta\hat{L}) - \sinh(\eta\hat{L}_z)\sinh(\eta(\hat{L}_z-1))]^{1/2}}{[\hat{L}^2 - \hat{L}_z^2 + \hat{L}_z]^{\frac{1}{2}}\sinh(\eta)}. \tag{14}$$

Substituting the expressions (2) into (12)–(14), we finally obtain a new multi-fermion realization of $\mathrm{SU}_q(2)$. In contrast to the realization given in [12], there are two useful features in (12)–(14). One is that $\hat{J}_\pm$ have been expressed as functions of $\hat{L}_z$ and $\hat{L}_\pm$. Another is that the symmetry among particles in $\hat{L}_z$ and $\hat{L}_\pm$ is preserved in $\hat{J}_z$ and $\hat{J}_\pm$.

With the help of (12)–(14), we can write the Hamiltonian (7) as

$$\hat{H}(\eta) = \varepsilon\hat{J}_z - \frac{V}{2}[G(\hat{L}, \hat{L}_z)\hat{L}_+^2 + \hat{L}_-^2 G(\hat{L}, \hat{L}_z)]. \tag{15}$$

where

$$G(\hat{L}, \hat{L}_z) = \phi(\hat{L}, \hat{L}_z - 1)\cdot\phi(\hat{L}, \hat{L}_z - 2) \tag{16}$$

with

$$\phi(\hat{L}, \hat{L}_z) = \frac{[\sinh^2(\eta\hat{L}) - \sinh(\eta\hat{L}_z)\sinh(\eta(\hat{L}_z+1))]^{1/2}}{[\hat{L}^2 - \hat{L}_z^2 - \hat{L}_z]^{1/2}\sinh(\eta)}. \tag{17}$$

Since $\hat{H}(\eta)$ is now expressed as a function of $\hat{L}_z$ and $\hat{L}_\pm$, like the original Hamiltonian (1), it can also be diagonalized in a subspace spanned by basic vectors

$$|n\rangle = \sqrt{\frac{(N-n)!}{n!N!}}(\hat{L}_+)^n \prod_{p=1}^{N} a_{p-}^+|0\rangle \qquad n = 0, 1, 2, \ldots, N \tag{18}$$

which are the common eigenstates of $\hat{L}^2$ and $\hat{L}_z$, i.e.

$$\hat{L}^2|n\rangle = \frac{N}{2}\left(\frac{N}{2}+1\right)|n\rangle \qquad \hat{L}_z|n\rangle = \left(-\frac{N}{2}+n\right)|n\rangle. \tag{19}$$

In contrast to [11], here we do not have to perform our calculations in the q-deformed representation. On the other hand, comparing (15) with (1), one notices that q-deformation induces only the operator function $G(\hat{L}, \hat{L}_z)$. Therefore, all the physical effects created by q-deformation come from the modification of $G(\hat{L}, \hat{L}_z)$ to the interaction strength. The matrix element of $\hat{H}(\eta)$ between the states $|n'\rangle$ and $|n\rangle$ $(n > n')$ can be written as

$$\langle n|\hat{H}(\eta)|n'\rangle = -\frac{V}{2}\langle n|G(\hat{L}, \hat{L}_z)|n\rangle\langle n|\hat{L}_+^2|n'\rangle. \tag{20}$$

We see that the interaction strength is modified by the factor $\langle n|G(\hat{L}, \hat{L}_z)|n\rangle$. The interaction will be enhanced if $\langle n|G(\hat{L}, \hat{L}_z)|n\rangle > 1$ for all states, and suppressed otherwise. We shall numerically show the behaviour of the factor $\langle n|G(\hat{L}, \hat{L}_z)|n\rangle$ in the following.

The Hartree–Fock (HF) approximation is extensively used in dealing with quantum many-body problems. The basic idea of the HF approximation is to choose a proper determinant composed of single particle states as the approximate ground state vector of a quantum many-body system. In (15), the interaction term represents the state-dependent monopole particle–hole excitation between the upper and lower levels. The operator $\hat{L}_+$ can create this excitation from the unperturbed ground state $\prod_{p=1}^{N} a_{p-}^{+}|0\rangle$. Therefore, instead of employing the q-coherent state as in [11], we here take the usual su(2) coherent state

$$|\Phi(z)\rangle = \langle\Phi(z)|\Phi(z)\rangle^{1/2} \exp(z\hat{L}_+) \prod_{p=1}^{N} a_{p-}^{+}|0\rangle \tag{21}$$

as the HF trial determinant, where the parameter $z = \rho\exp(\mathrm{i}\phi)$ is a complex number. The expectation value of $\hat{H}(\eta)$ in (21) can be found as

$$\begin{aligned} E(\eta,\rho,\phi) = &-\frac{1}{2}\varepsilon N + \frac{\rho^2}{1+\rho^2}\varepsilon N \\ &-V\sum_{k=0}^{N-2}\frac{N!}{k!(N-k-2)!}S(\eta,k)S(\eta,k+1)\frac{\rho^{2k+2}}{1+\rho^2}\cos 2\phi \end{aligned} \tag{22}$$

where

$$S(\eta,k) = \frac{\left[\sinh^2\left(\eta\sqrt{\frac{N}{2}(\frac{N}{2}+1)}\right) - \sinh(\eta(\frac{N}{2}-k))\sinh(\eta(\frac{N}{2}-k-1))\right]^{1/2}}{[\frac{N}{2}(\frac{N}{2}+1)-(\frac{N}{2}-k)^2+(\frac{N}{2}-k)]^{1/2}\sinh(\eta)}. \tag{23}$$

The ground state energy under the HF approximation is the minimal value of $E(\eta,\rho,\phi)$ with respect to ρ and ϕ. In (22), the variation region of ρ is from 0 to $+\infty$ and ϕ from 0 to 2π. We see in (22) that the minimal point must be at $\phi = 0$ and $\phi = \pi$. Although the values of $E(\eta,\rho,\phi)$ at $\phi = 0$ and π are the same, the corresponding state vectors (21) are different: $\exp(\rho\hat{L}_+)\prod_{p=1}^{N} a_{p-}^{+}|0\rangle$ for $\phi = 0$ and $\exp(-\rho\hat{L}_+)\prod_{p=1}^{N} a_{p-}^{+}|0\rangle$ for $\phi = \pi$. To consider equally the states we extend the variation region of ρ to $-\infty$ while applying the restriction $\phi = 0$. If the parameter $\eta = 0$, using (22) it is easily shown that when $VN/\varepsilon < 1$ the minimal point is at $\rho = 0$, but when $VN/\varepsilon > 1$ there are two minimal points symmetrically about $\rho = 0$ [16, 17]. The excitation spectrum and the ground state properties of the system in the two interaction regions are very different. As usual, we say that the system undergoes a ground state phase transition when VN/ε changes from values smaller than unity to larger than unity. Here, we are interested in the influence of q-deformation on the phase transition.

In figure 1, the modifying factors $\langle n|G(\hat{L},\hat{L}_z)|n\rangle$ of the interaction for the various states are depicted. One notices that the values of all the factors are larger than unity. In fact, we have found that their values are always larger than unity as long as $\eta \neq 0$, and increase rapidly with η increasing. As mentioned in the above, the monopole particle–hole interaction is strengthened by q-deformation. Therefore, we can conclude that the ground state phase transition must be enhanced in the q-deformed LMG model. It should be pointed out that the conclusion obtained in [11] is not in agreement with the present result. We notice that the parameter $\chi = V[N]/\varepsilon$ with $[N] = (q^N - q^{-N})/(q - q^{-1}) = \sinh(\eta N)/\sinh(\eta)$ was used in [11]. Because $[N]$ increases monotonically with q or η increasing for a fixed N, it is equivalent to decreasing the interaction strength V if one fixes χ but increases q or η. Therefore, suppression of the ground state phase transition in the q-deformed LMG model, which was declared in [11], is simply induced by decreasing the interaction strength.

In figure 2, the exact excitation energies of the first several eigenstates of $\hat{H}(\eta)$ are plotted against the parameter VN/ε. In figure 2(a), we observe that when $VN/\varepsilon < 1$ the excitation spectrum is nearly a typical vibration spectrum with unity space, and when $VN/\varepsilon > 1$ the lower lying states become double degenerate in energy. Comparing

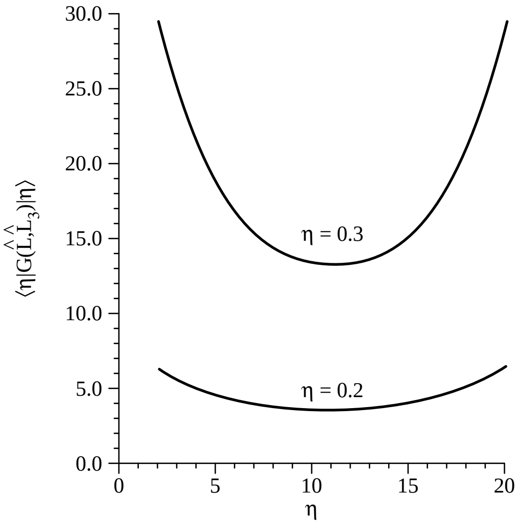


Figure 1. The matrix elements $\langle n|G(\hat{L}, \hat{L}_z)|n\rangle$ for $N = 20$.

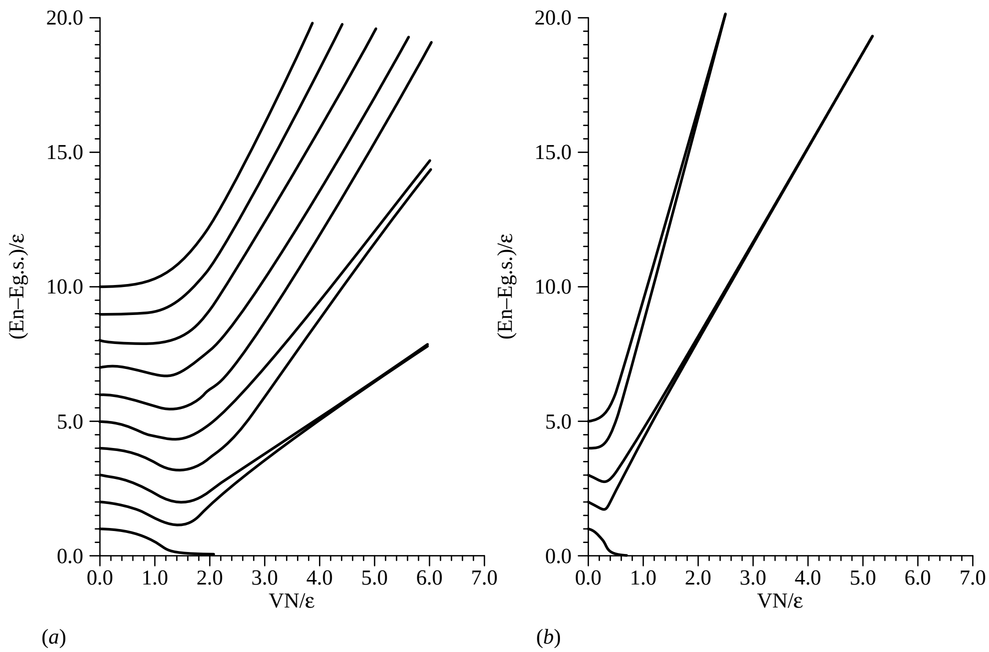


Figure 2. The excitation energies versus VN/ε for $N = 20$. (a) $\eta = 0.0$; (b) $\eta = 0.2$.

figures 2(a) and (b), we see that when $\eta \neq 0$ the range of values of VN/ε for the vibration spectrum becomes much smaller and the double degeneracy appears even for very small values of $VN/\varepsilon(< 1)$. This means that phase transition is enhanced by the q-deformation, which is just as we expect. One of the characteristics of the ground state phase transition is the emergence of the zero-frequency vibration mode. The first excitation energy versus VN/ε is shown in figure 3. It is noticed that in the case $\eta = 0$ the energy approaches zero as VN/ε increases, but when $\eta \neq 0$ the energy initially sharply decreases and then gradually increases.

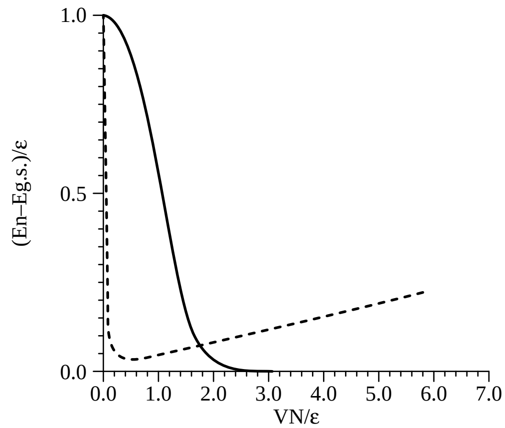
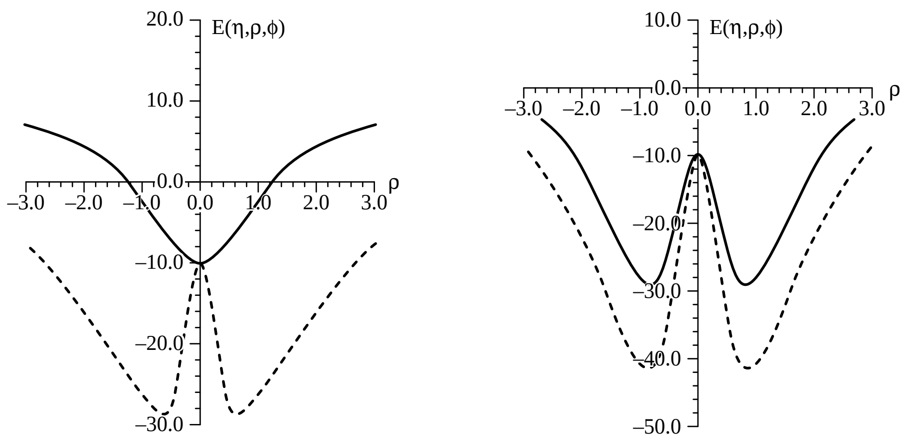


Figure 3. The first excitation energy versus VN/ε for $N = 20$ with $\eta = 0.0$ (full curve) and with $\eta = 0.35$ (broken curve).



(*a*) (*b*)

Figure 4. The energy surface $E(\eta, \rho, \phi)$ as a function of ρ for $N = 20$. (*a*) $VN/\varepsilon = 0.4$, $\eta = 0$ (full curve) and $\eta = 0.3$ (broken curve); (*b*) $VN/\varepsilon = 6.0$, $\eta = 0$ (full curve) and $\eta = 0.1$ (broken curve).

The changes of excitation spectrum observed may be understood according to the behaviour of $E(\eta, \rho, \phi)$ with respect to ρ. In figure 4, we show a cut ($\phi = 0$) through the two-dimensional energy surface (22) for different values of VN/ε and η. Obviously, if the minimum appears at $\rho = 0$ the excitation spectrum must exhibit the behaviour of an equal-spaced vibration spectrum. And if two minima appear symmetrically about the original, states which have exciting energies lower than the barrier between the two minimal points must be double degenerate in energy. If $\eta = 0$, the first and second cases happen when $VN/\varepsilon < 1$ and $VN/\varepsilon > 1$, respectively. If $\eta \neq 0$, we notice that even for very small

$VN/\varepsilon(<1)$ the point $\rho = 0$ becomes unstable and two symmetrical minimal points appear as shown in figure 4(a). In comparison with the case of $\eta = 0$, as shown in figure 4(b), the minimal points become lower and the barrier between the minimal points is narrower when $\eta \neq 0$. In our calculations we have found that these changes are more pronounced as VN/ε becomes larger. Since the barrier becomes much narrower and the tunnel effect is more pronounced as VN/ε increases, the first double degeneracy is dismissed and the first excitation energy becomes non-zero as shown in figure 3.

Table 1. Comparison of the exact ground state energy $E^{\text{ex.}}_{\text{g.s.}}/\varepsilon$ with the HF approximation $E^{\text{HF}}_{\text{g.s.}}/\varepsilon$.

	N	10		20		40	
η	VN/ε	$E^{\text{ex.}}_{\text{g.s.}}$	$E^{\text{HF}}_{\text{g.s.}}$	$E^{\text{ex.}}_{\text{g.s.}}$	$E^{\text{HF}}_{\text{g.s.}}$	$E^{\text{ex.}}_{\text{g.s.}}$	$E^{\text{HF}}_{\text{g.s.}}$
	0.0	−5.0000	−5.0000	−10.0000	−10.0000	−20.0000	−20.0000
	0.4	−5.0366	−5.0000	−10.0391	−10.0000	−20.0404	−20.0000
	0.6	−5.0839	−5.0000	−10.0912	−10.0000	−20.0954	−20.0000
0.0	0.8	−5.1533	−5.0000	−10.1717	−10.0000	−20.1838	−20.0000
	1.0	−5.2480	−5.0000	−10.2915	−10.0000	−20.3288	−20.0000
	2.0	−6.2235	−5.8887	−12.3242	−12.1314	−24.7902	−24.6280
	5.0	−12.1030	−11.8055	−25.0558	−24.8024	−51.0452	−50.8010
	0.0	−5.0000	−5.0000	−10.0000	−10.0000	−20.0000	−20.0000
	0.4	−5.0488	−5.0000	−10.1254	−10.0000	−23.6737	−23.5316
	0.6	−5.1123	−5.0000	−10.3085	−10.0000	−28.6627	−28.4887
0.1	0.8	−5.2058	−5.0000	−10.6269	−10.2580	−34.3369	−34.0929
	1.0	−5.3338	−5.0032	−11.1486	−10.8602	−40.3749	−40.0456
	2.0	−6.5989	−6.2778	−16.0944	−15.8706	−72.8356	−71.9681
	5.0	−13.3263	−12.9796	−35.5783	−35.0639	−175.4672	−172.8557
	0.0	−5.0000	−5.0000	−10.0000	−10.0000	−20.0000	−20.0000
	0.4	−5.1075	−5.0000	−11.9587	−11.6940	−222.0260	−216.6620
	0.6	−5.2487	−5.0000	−14.4736	−14.2396	−330.8338	−322.1046
0.2	0.8	−5.4573	−5.0729	−17.4264	−17.1420	−440.0223	−427.9205
	1.0	−5.7393	−5.3330	−20.5955	−20.2315	−549.3819	−533.9264
	2.0	−8.0660	−7.7362	−37.6774	−36.8084	−1097.0073	−1065.0435
	5.0	−17.6452	−17.1052	−91.4588	−89.0265	−2741.2701	−2660.4930

The ground state energy under the HF approximation $E^{\text{HF}}_{\text{g.s.}}/\varepsilon$ is compared with the exact one $E^{\text{ex.}}_{\text{g.s.}}/\varepsilon$ for different values of the q-deformation parameter η in table 1. We see that the HF approximation on the basis of the usual su(2) coherent state is also very well to the q-deformed system. In table 1, the results with a fixed N but different η clearly show the enhancement of the ground state phase transition in the q-deformed LMG model. We also notice that the enhancement effect is more pronounced as N becomes large.

In summary, we find a new multi-fermion realization of quantum algebra $\mathrm{SU}_q(2)$, whose generators have been written as functions of the operators of su(2) algebra. We show that the interaction is strengthened by q-deformation and that the ground state phase transition is enhanced in the q-deformed LMG model. We also show that the HF approximation on the basis of the usual su(2) coherent state still works well for the q-deformed system.

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